Journal of Applied Mathematics and Mechanics 73 (2009) 88–102

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

A non-equilibrium model of a porous medium saturated with immiscible fluids ☆

V.I. Kondaurov

Dolgoprudnyi, Russia

ARTICLE INFO

Article history:
Received 29 January 2008

ABSTRACT

A model of a thermoelastic porous medium, saturated with two immiscible fluids, is considered. It is assumed that there are no phase transitions, the contribution of pulsations to the stress and kinetic energy is small, and that the components of the medium are in thermodynamic equilibrium. The non-equilibrium of the state, related to the finite time of redistribution of the fluids among the pores of the channels due to the presence of surface forces, is taken into account. A general form of the governing relations, necessary and sufficient to satisfy the principles of thermodynamic compatibility and independence of the choice of system of coordinates, is obtained. It is shown that the establishment of equilibrium is accompanied by dissipation due to capillary forces, which does not lead to seepage dissipation or thermal dissipation. For the case when the deformation of the skeleton and the deviation of the mean porous pressure and the temperature from the initial values are small, while the saturation and the non-equilibrium parameter undergo finite changes, an approximation of the potential of the skeleton is proposed in the form of a quadratic expansion in small parameters. A feature of the expansion is the presence of an initial value of the potential, which depends on the saturation and non-equilibrium. The relationship between the thermodynamic potential and the non-equilibrium kinetics, related to the requirement that the dissipation by the capillary forces should be non-negative, is determined. A generalized Darcy's law is formulated, which takes cross terms into account. It is shown that the proposed approximations enable key effects, which accompany the motion of immiscible fluids in a porous medium, to be described.

1. Introduction

Many natural, biological and technological processes involve the seepage of immiscible fluids in a porous medium. Typical examples are petroleum and water in oil-bearing strata, water and moist air in the ground and in the wood. Such media possess behavioural features that are unusual for a porous medium, saturated solely with a single fluid. An example of macroscopic effects are the reduction in the size of the body when drying, the rise of a fluid on saturation, and spontaneous displacement of a wetting fluid by another, less wetting fluid. It is clear intuitively that all these phenomena are related to surface tension. The principles of the theory of the seepage of immiscible fluids has been described in classical publications[1–3] and developed in numerous subsequent papers.

A noteworthy feature of a porous medium, saturated with immiscible fluids, is the fact that the process depends on the rate and direction of the change of state. The most well-known and often discussed phenomena of this type are the relaxation of capillary pressure, the "capillary pressure – saturation" hysteresis curve, and the dependence of the phase permeabilities and the value of the capillary sticking on the rate and direction of a change in the saturation. The generally accepted explanation of these phenomena is the non-equilibrium of the common motion of the fluids.[4,5] If there is no equilibrium, the change in the saturation of the material element (or any other action) leads to a situation where the state of the element later also changes, even when there is no mass, momentum and energy exchange with the surrounding particles of the body. This is due to the fact that, after a finite (or infinite) time, the fluids are redistributed between the porous channels. As a result, the less-wetting fluid occupies the coarse capillaries and the wetting fluid occupies the fine capillaries. The reason for this is a reduction in the surface energy of the element of the porous medium.

In the model considered, the square of the non-equilibrium parameter is identified with twice the surface energy of contact between the non-wetting fluid and the skeleton. The energy of the surfaces, separating the fluids, is neglected, since the area of these surfaces is

---

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 1, pp. 121-142, 2009.

E-mail address: vk@mipt.ru.

0021-8928/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2009.03.004

small compared with the area of contact of the fluids with the skeleton. This assumption is analogous to neglecting the interaction between the flows in the capillary channels (molecular and pulsation exchange with pulses), whereas the viscous interaction of the fluids with the skeleton is assumed to be the most important factor. Speaking non-rigorously, each fluid from the macroscopic point of view is assumed to be an "ideal" gas, the particles of which do not interact with one another, but momentum and energy are exchanged with the skeleton. This implicit assumption is typical for models of porous media, in which the effect of the viscous (and even viscoplastic) properties of the fluids on the stress state is neglected, but their effect when interacting with the skeleton is taken into account.[6,7]

This interpretation of the non-equilibrium parameter enables us to derive a kinetic equation for it in the form of an ordinary differential equation along the trajectory of the particles of the skeleton. The use of kinetics, rather than some effective viscosity, is due to two reasons. First, the establishment of equilibrium after a finite time denotes that the material must possess a long memory on previous states, rather than instantaneous memory, inherent in viscous fluids. Second, the redistribution time depends on the viscosities of the interacting fluids and is accompanied by energy dissipation, which is non-zero even if the macroscopic velocities are equal to zero. This means that in addition to the seepage dissipation, related to the viscous friction of the fluids on the skeleton and heat dissipation due to heat fluxes, the transition to the equilibrium state must be characterized by a new form of energy dissipation. This form of dissipation is called dissipation by capillary forces.

The question of the kinetics of the non-equilibrium remains open to a considerable extent. Thus, for example, it was assumed in Ref. 5 that the rate at which the system tends towards equilibrium is proportional to the difference between the actual and effective saturation, which define the current seepage properties of the medium. This simplification enables a number of non-equilibrium effects to be modelled, but the thermodynamics of these phenomena have not been considered.

Some progress in this direction has been obtained by using a general form of the governing relations, necessary and sufficient to satisfy the principles of thermodynamic compatibility and independence of the choice of the reference frame. As previously,[8] when deriving the general form of the constitutive relations, a representation of the forces of interaction of the components of the medium in the form of the sum of the equilibrium and dissipative components is used. A governing relation is postulated for each of the dissipative terms, whereas the equilibrium components are described by universal expressions, independent of the rheological properties of the fluid. Expansion of the interaction forces leads to a new form of entropy inequality, which enables a number of consequences to be obtained, some of which were known previously either in the form of sufficient conditions or in the form of additional assumptions.

The general form obtained shows that the dissipation by the capillary forces is equal to the product of the generalized force (the derivative of the thermodynamic potential of the skeleton with respect to the non-equilibrium parameter) and the generalized velocity (the kinetic function), each of them being an alternating-sign function. The simplest method of satisfying the requirement that this product should be non-negative is to assume that the rate of change of the non-equilibrium parameter is proportional to the derivative of the potential of the skeleton with respect to this parameter. The coefficient of proportionality can be any positive function, depending on the state parameters. Other methods of obtaining the kinetic equation, satisfying the condition for the dissipation by the capillary forces to be non-negative have not so far been examined. Hence, the new idea of the thermodynamics of multiphase seepage – the dissipation by capillary forces – reveals the interconnection between the thermodynamic potential and non-equilibrium kinetics.

For the case when the deformation of the skeleton and the deviation of the pore pressure and temperature from the initial values are small, while the saturation and non-equilibrium parameter undergo finite changes, the simplest approximation of the thermodynamic potential of the skeleton in the form of a quadratic expansion in small parameters is proposed. A feature of the expansion is the presence of an initial value of the thermodynamic potential. In the classical mechanics of a homogeneous medium, the initial value of the potential, without loss of generality, can be taken to be zero. But, if there are transformations, accompanied by a transition of the thermomechanical forms of energy into "latent" energy of transformation, the initial value of the potential is non-zero and is an energy measure of the depth of the transformation.[9] A similar situation occurs for a porous medium saturated with several fluids. The energy of the surface of contact of the less wetting fluid with the skeleton plays the role of the latent energy in this case. This energy is equal to zero if the element of the porous medium in the reference state is completely saturated with the wetting fluid, while for partial saturation the initial value of the potential is positive.

The construction of a quasi-linear approximation for the dissipative interaction forces led to a generalized Darcy's law. The difference from the classical models is due to the fact that cross terms, which denote the dependence of the seepage vector on the pressure gradient in each fluid, are taken into account. Neglect of these terms is based on the implicit assumption that the fluids are separated in the element of the porous medium, and each fluid moves along its own capillaries. In the non-equilibrium state no such separation occurs and it is necessary to take their interaction into account. Moreover, it seams that the cross terms depend very much on the degree of non-equilibrium, vanishing when equilibrium is reached.

On the basis of the representations developed, we consider examples which show that even the simplest approximations enable one to describe the above-mentioned key effects. It is also shown that non-equilibrium affects the Buckley function, which specifies the fraction of the wetting fluid in the total flow that, in turn, leads to a considerable difference between the seeping flows, obtained using the classical models and using the approach developed below.

2. Kinematics and conservation laws

The assumption that saturated immiscible fluids in a porous medium are a set of mutually penetrating and mutually interacting continua, simultaneously occupying the same volume considered, is the initial premise. In this case we consider three continua – the solid skeleton, denoted by the subscript s, and two fluids, denoted by the Greek subscript $\alpha = 1, 2$. Scalar volume fractions ϕ_A, $A = s, \alpha$ are ascribed to these continua, so that $\Sigma\phi_A = 1$. The sum of the volume fractions of the fluids is equal to the porosity $\phi = \phi_1 + \phi_2$. The saturation of the porous medium by the first fluid is $S = \phi_1/\phi$ and the saturation by the second fluid is equal to $1 - S = \phi_2/\phi$. The equalities $\phi_1 = S\phi$, $\phi_2 = (1 - S)\phi$ hold for the volume fractions of each fluid.

Suppose χ is a fixed region in Euclidean space. The material elements (particles) of the skeleton and the fluids are matched at each point $\boldsymbol{x} \in \xi$ of this region. We denote by $\kappa_A(t)$ the regions of space which are occupied at the initial instant by particles of the continuum $A = s, \alpha$, which are in the region χ at the current instant of time t. The regions $\kappa_A(t)$ will be called the initial configurations of the skeleton $(A = s)$

and the fluids ($A = d$). The time dependence of the region $\kappa_A(t)$ denotes that different sets of particles of the skeleton and the fluids are in the region χ at different instants of time.

The mappings $\kappa_A(t) \to \chi$ are assumed to be piecewise one-to-one and differentiable. The laws of motion then possess the same properties:

$$\mathbf{X}_A = \mathbf{X}_A(\mathbf{x}, t), \quad \mathbf{x} \in \chi, \quad \mathbf{X}_A \in \kappa_A(t), \quad t \geq 0, \quad A = s, \alpha \tag{2.1}$$

Suppose $\mathbf{G}_A = [\nabla \otimes \mathbf{X}_A(\mathbf{x}, t)]^T$ is the distortion of the continuum A, so that $d\mathbf{X}_A = \mathbf{G}_A \cdot d\mathbf{x}$. It follows from the uniqueness of the mappings (2.1) that $J_A = \det \mathbf{G}_A \neq 0$. For a non-degenerate tensor we have the polar expansion

$$\mathbf{G}_A = \mathbf{R}_A \cdot \mathbf{U}_A = \mathbf{V}_A \cdot \mathbf{R}_A \tag{2.2}$$

where $\mathbf{U}_\mathrm{A}$ and $\mathbf{V}_\mathrm{A}$ are symmetrical positive-definite tension (compression) tensors, and $\mathbf{R}_\mathrm{A}$ is the orthogonal rotation tensor. Expansions (2.2) possess the property of uniqueness.

Suppose further that $\boldsymbol{\upsilon}_A = \partial\mathbf{x}(\mathbf{X}_A, t)/\partial t|_{\mathbf{X}_A}$ and $\mathbf{M}_A = \mathbf{X}_A(\mathbf{x}, t)/\partial t|_{\mathbf{x}}$ is the mass velocity and the velocity of displacement of the particles of the continuum A at a specified point $\mathbf{x}$. Differentiating the motion laws (2.1), written in the form of the complex function $\mathbf{X}_A = \mathbf{X}_A(\mathbf{x}(\mathbf{X}_A, t), t$, with respect to time for a fixed particle $\mathbf{X}_\mathrm{A}$, we arrive at the relation $\mathbf{M}_A = -\mathbf{G}_A \cdot \boldsymbol{\upsilon}_A$, which connects the velocity with the known distortion.

The distortion $\mathbf{G}_\mathrm{A}(\mathbf{x},\mathrm{t})$ and the velocity vector $\mathbf{M}_\mathrm{A}(\mathbf{x},\mathrm{t})$ are the first derivatives of the mapping (2.1). There is therefore a relation between them, namely, the equation of compatibility of the deformations and velocities[9,10]

$$d_A \mathbf{G}_A / dt = -\mathbf{G}_A \cdot (\nabla \otimes \boldsymbol{\upsilon}_A)^T \tag{2.3}$$

where $d_A f/dt = \partial f(\mathbf{x}, t)/\partial t|_{\mathbf{x}} + \boldsymbol{\upsilon}_A \cdot \nabla f$ is a derivative along the trajectory of a particle of the continuum A. Summation is not carried out over the index A, unless otherwise stated.

We will confine ourselves to the case when there is no mass exchange between the continua (a phase or chemical transition). Then the law of conservation of mass for each continuum can be written in the form

$$r_A = r_A^0 J_A; \quad H_A = \det \mathbf{G}_A, \quad A = s, \alpha \tag{2.4}$$

where r_A and r_A^0 are the average density of the mass of the continuum A in the current and initial states. From relation (2.4) we obtain the equation of continuity

$$\partial r_A / \partial t|_{\mathbf{x}} + \nabla \cdot (r_A \boldsymbol{\upsilon}_A) = 0 \tag{2.5}$$

It is assumed that a force $\mathbf{f}_\mathrm{A}$, equal to the sum of the external mass force $\mathbf{f}_A^b$, the interaction force $\mathbf{f}_A^{\mathrm{int}}$ and the contact force $\mathbf{f}_A^c$, acts on the continuum A in the region χ

$$\mathbf{f}_A = \mathbf{f}_A^b + \mathbf{f}_A^{\mathrm{int}} + \mathbf{f}_A^c = \int_\chi r_A \mathbf{g} dV - \int_\chi \mathbf{b}_A^{\mathrm{int}} dV + \oint_{\partial\chi} \mathbf{t}_A dS$$

where $\mathbf{g}(\mathbf{x},\mathrm{t})$ is the density of the external mass force (the gravity force), $\mathbf{t}_\mathrm{A}$ is the stress vector (the force per unit area of the surface $\partial\chi$) and $\mathbf{b}_A^{\mathrm{int}}$ is the specific force of interaction of the continuum A with other continua, comprising the medium considered.

It is assumed that the sum of the interaction forces of the continua are equal to zero:

$$\sum \mathbf{b}_A^{\mathrm{int}} = \mathbf{b}_s^{\mathrm{int}} + \mathbf{b}_1^{\mathrm{int}} + \mathbf{b}_2^{\mathrm{int}} = 0 \tag{2.6}$$

Assumption (2.6) can be validated by averaging the law of conservation of momentum for an elementary volume containing a large number of pores filled with fluids.[11]

The relation of the forces acting on the continuum A to the motion of this continuum occurs via the laws of conservation of momentum and angular momentum. If the continua are non-polar and the Cauchy postulate $\mathbf{t}_\mathrm{A} = \mathbf{t}_\mathrm{A}(\mathbf{x},\mathbf{n})$ holds, then $\mathbf{t}_A(\mathbf{x}, \mathbf{n}) = \mathbf{T}_A(\mathbf{x}) \cdot \mathbf{n}$, where $\mathbf{T}_\mathrm{A}$ is the tensor of the averaged partial stresses of the continuum A. The symmetry $\mathbf{T}_A^T = \mathbf{T}_A$ of the partial-stress tensor follows from the law of conservation of angular momentum.

For smooth motion the equation of the local momentum balance of continuum A can be written in the form

$$\partial(r_A \boldsymbol{\upsilon}_A)/\partial t|_{\mathbf{x}} + \nabla \cdot (r_A \boldsymbol{\upsilon}_A \otimes \boldsymbol{\upsilon}_A - \mathbf{T}_A^T) = r_A \mathbf{g} - \mathbf{b}_A^{\mathrm{int}}$$

Taking Eq. (2.5) into account, this relation gives the equation of motion

$$r_A d_A \boldsymbol{\upsilon}_A / dt - \nabla \cdot \mathbf{T}_A = r_A \mathbf{g} - \mathbf{b}_A^{\mathrm{int}} \tag{2.7}$$

The interaction forces, the partial stresses and the volume fractions of the fluids can be represented in the form of the sum equilibrium and dissipative components

$$\mathbf{b}_\alpha^{\mathrm{int}} = \mathbf{b}_\alpha^{\mathrm{eq}} + \mathbf{b}_\alpha^{\mathrm{dis}}, \quad \mathbf{T}_\alpha = \mathbf{T}_\alpha^{\mathrm{eq}} + \mathbf{T}_\alpha^{\mathrm{dis}}, \quad \phi_\alpha = \phi_\alpha^{\mathrm{eq}} + \phi_\alpha^{\mathrm{dis}} \tag{2.8}$$

the dissipative components $\mathbf{b}_\alpha^{\mathrm{dis}}, \mathbf{T}_\alpha^{\mathrm{dis}}, \phi_\alpha^{\mathrm{dis}}$ vanish in the equilibrium state, in which the temperature gradient and the mass velocities are equal to zero in a certain neighbourhood of the current instant of time. The equilibrium force of interaction of the fluid α with the other continua is related to the true stress $\boldsymbol{\sigma}_\alpha^{eq}$ and the gradient of the equilibrium volume fraction $\nabla\phi_\alpha^{\mathrm{eq}}$ as follows:

$$\mathbf{b}_\alpha^{\mathrm{eq}} = \boldsymbol{\sigma}_\alpha^{\mathrm{eq}} \cdot \nabla \phi_\alpha^{\mathrm{eq}} \tag{2.9}$$

For an equilibrium state the accelerations, in fact, are equal to zero. Hence, the equation of motion (2.7) reduces to the equilibrium equation $\nabla\cdot\mathbf{T}_\alpha^{\mathrm{eq}} = -r_\alpha^{\mathrm{eq}}\mathbf{g} + \mathbf{b}_\alpha^{\mathrm{eq}}$, where $r_\alpha^{\mathrm{eq}} = \phi_\alpha^{\mathrm{eq}}\rho_\alpha^{\mathrm{eq}}$ is the average equilibrium density. Using the relation between the average and true stresses $\mathbf{T}_\alpha^{\mathrm{eq}} = \phi_\alpha^{\mathrm{eq}}\boldsymbol{\sigma}_\alpha^{\mathrm{eq}}$, the equilibrium equation can be written in the form

$$\phi_\alpha^{\mathrm{eq}}\nabla\cdot\boldsymbol{\sigma}_\alpha^{\mathrm{eq}} + \boldsymbol{\sigma}_\alpha^{\mathrm{eq}}\cdot\nabla\phi_\alpha^{\mathrm{eq}} = -\phi_\alpha^{\mathrm{eq}}\rho_\alpha^{\mathrm{eq}}\mathbf{g} + \mathbf{b}_\alpha^{\mathrm{eq}}$$

In a state of rest $\nabla\cdot\boldsymbol{\sigma}_\alpha^{\mathrm{eq}} + \rho_\alpha^{\mathrm{eq}}\mathbf{g} = 0$. Hence formula (2.9) follows.

Expression (2.9) holds for a fluid with any rheological properties, unlike the dissipative term $\mathbf{b}_\alpha^{\mathrm{dis}}$, for which it is necessary to formulate a governing relation, which leads to a generalized Darcy's law.

We will assume that the porous medium is in local thermal equilibrium, i.e., the elements of the skeleton and the fluids at a point **x** are characterized by a single absolute temperature $\theta(\mathbf{x}, t) > 0$, which may vary with time and from point to point. The first law of non-equilibrium thermodynamics for smooth motions of a porous medium in this approximation is postulated in the form of the energy balance equation

$$\partial\sum r_a e_A/\partial t + \nabla\cdot\partial(r_A e_A\boldsymbol{\upsilon}_A - \mathbf{T}_A\cdot\boldsymbol{\upsilon}_A) - \nabla\cdot\mathbf{q} = \sum r_a\mathbf{g}\cdot\boldsymbol{\upsilon}_A + rQ$$

where $e_A = u_A + v_A\cdot v_A/2$ is the specific total energy of the continuum A, u_A is the specific internal energy, $\mathbf{q}$ is the total thermal flux, Q is the density of the heat sources and $r = \Sigma r_A$ is the total mass density. This form of writing the total energy is approximate since it does not take into account the contribution of pulsations to the kinetic energy[11] and the effect of the crookedness of the porous channels.[4]

When the equation of continuity (2.5), the equation of motion (2.7) and relation (2.6) are taken into account, the local law of conservation of energy reduces to the form

$$\sum_A r_A d_A u_A/dt = \sum_A \mathbf{T}_A : (\nabla\otimes\boldsymbol{\upsilon}_A) + \sum_\alpha \mathbf{b}_\alpha^{\mathrm{bit}}\cdot\mathbf{w}_\alpha + \nabla\cdot\mathbf{q} + rQ \tag{2.10}$$

Here and below $\mathbf{w}_\alpha = v_\sigma - v_s$ is the velocity of motion of the fluid α with respect to the skeleton.

The second law of non-equilibrium thermodynamics for smooth motions of a porous medium, which is saturated with two fluids and is characterized by a temperature that is common for all the continua, is formulated in the form of a local inequality for the entropy production rate

$$\partial\sum r_A\eta_A/\partial t + \nabla\cdot\sum r_A\eta_A\boldsymbol{\upsilon}_A - \nabla\cdot(\mathbf{q}/\theta) - rQ/\theta \ge 0$$

where η_A is the specific entropy of the continuum A. When the equations of continuity (2.5) are taken into account, this relation can be written in the form of the inequality

$$\sum r_A d_A\eta_A/dt - \nabla\cdot(\mathbf{q}/\theta) - rQ/\theta \ge 0 \tag{2.11}$$

Inequality (2.11) is regarded as the condition which the governing relations must satisfy for any histories of the change of state of the medium considered.[12,13] To derive the limitations on the governing relations we can use the so-called reduced form of the inequality for the entropy production rate, which does not contain the heat supply rate:

$$-\sum_A\left(r_A\frac{d_A\psi_A}{dt} + r_A\eta_A\frac{d_A\theta}{dt}\right) - (\mathbf{T}\cdot\mathbf{G}_s^{-1}) : \frac{d_s\mathbf{G}_s^T}{dt} + \sum_\alpha\boldsymbol{\sigma}_\alpha^{\mathrm{eq}} : (\nabla\otimes(\phi_\alpha^{\mathrm{eq}}\mathbf{w}_\alpha)) + \delta_f + \delta_T \ge 0 \tag{2.12}$$

where $\mathbf{T} = \Sigma\mathbf{T}_A$ is the total stress tensor, $\delta_T = \theta^{-1}\mathbf{q}\cdot\nabla\theta$ is the thermal dissipation and $\delta_f = \sum_\alpha\mathbf{T}_\alpha^{\mathrm{dis}} : (\nabla\otimes\omega_\alpha) + \mathbf{b}_\alpha^{\mathrm{dis}}\cdot\mathbf{w}_\alpha$ is the seepage dissipation.

To derive inequality (2.12) we use Eqs (2.3) and (2.9). This gives the relation

$$\sum_A(\mathbf{T}_A : (\nabla\otimes\boldsymbol{\upsilon}_A)) + \sum_\alpha\mathbf{b}_\alpha^{\mathrm{dis}}\cdot\mathbf{w}_\alpha = -(\mathbf{T}\cdot\mathbf{G}_s^{-1}) : d_s\mathbf{G}_s^T/dt + \sum_\alpha\boldsymbol{\sigma}_\alpha^{\mathrm{eq}} : (\nabla\otimes(\phi_\alpha^{\mathrm{eq}}\mathbf{w}_\alpha)) + \delta_f \tag{2.13}$$

Substituting this relation into Eq. (2.10) of the local energy balance, we arrive at an expression for the of heat supply rate

$$\nabla\cdot\mathbf{q} + rQ = \sum_A r_A d_A u_A/dt + (\mathbf{T}\cdot\mathbf{G}_s^{-1}) : d_s\mathbf{G}_s^T/dt - \sum_\alpha\boldsymbol{\sigma}_\alpha^{\mathrm{eq}} : (\nabla\otimes(\phi_\alpha^{\mathrm{eq}}\mathbf{w}_\alpha)) - \delta_f$$

Substituting this relation into inequality (2.11) and using the expressions for the free-energy density $\psi_A = u_A - \theta\eta_A$ we obtain inequality (2.12).

3. The governing equations

The following assumptions are the basis of the non-equilibrium model of a porous medium, saturated with two immiscible fluids: each continuum possesses an instantaneous thermoelastic response, the medium is capable of conducting heat, the interaction forces depend on the relative velocities of motion, and the state of the material is also characterized by the degree of non-equilibrium.

These qualitative assertions are formalized as follows. The state of a point **x** at the instant of time t in the medium considered is specified by the following set of quantities

$$\Lambda(\mathbf{x}, t) = \{\mathbf{G}_A, \theta, \xi, \boldsymbol{\gamma}, \mathbf{w}_\alpha\}, \quad A = s, \alpha, \quad \alpha = 1, 2 \tag{3.1}$$

where ξ is a scalar non-equilibrium parameter and $\gamma \equiv \nabla\theta$ is the temperature gradient. The reaction (response) of an element of the medium at the point ($\mathbf{x}$, t) is characterized by the following set of quantities

$$\Upsilon(\mathbf{x}, t) = \{\psi_A(\mathbf{x}, t), \mathbf{T}_A(\mathbf{x}, t), \eta_A(\mathbf{x}, t), \phi_\alpha(\mathbf{x}, t), \mathbf{q}(\mathbf{x}, t), \mathbf{b}_\alpha^{\text{dis}}(\mathbf{x}, t)\}, \quad A = s, \alpha$$

which are assumed to be functions of the state

$$\Upsilon(\mathbf{x}, t) = \Upsilon^+\{\Lambda(\mathbf{x}, t)\}$$

where $\boldsymbol{\gamma}^+ = \{\psi_A^+, \mathbf{T}_A^+, \eta_A^+, \phi_\alpha^+, \mathbf{q}^+, \mathbf{b}_\alpha^+\}$ is a set of functions specifying the governing equations. The superscript "plus" (or other symbols) is used so as to distinguish functions from their values. These relations can be written in expanded form as follows:

$$\psi_A(\mathbf{x}, t) = \psi_A^+\{\Lambda(\mathbf{x}, t)\}, \quad \mathbf{T}_A(\mathbf{x}, t) = \mathbf{T}_A^+\{\Lambda(\mathbf{x}, t)\}, \quad \eta_A(\mathbf{x}, t) = \eta_A^+\{\Lambda(\mathbf{x}, t)\}$$
$$\phi_\alpha(\mathbf{x}, t) = \phi_\alpha^+\{\Lambda(\mathbf{x}, t)\}, \quad \mathbf{q}(\mathbf{x}, t) = \mathbf{q}^+\{\Lambda(\mathbf{x}, t)\}, \quad \mathbf{b}_\alpha^{\text{dis}}(\mathbf{x}, t) = \mathbf{b}_\alpha^+\{\Lambda(\mathbf{x}, t)\} \tag{3.2}$$

For the non-equilibrium parameter we have the following kinetic equation

$$d_s\xi/dt = \Omega^+\{\Lambda(\mathbf{x}, t)\} \tag{3.3}$$

The solution of kinetic equation (3.3) is a functional, defined by the prehistory of the quantities $\{\mathbf{G}_A(\tau), \theta(\tau), \boldsymbol{\gamma}(\tau), \mathbf{w}_\alpha(\tau)\}$, $\tau \leq t$ at the point $\mathbf{x}$. Hence, the material possesses a prolonged memory, while the parameter ξ itself reflects, in convoluted form, the effect of the history of the mechanical and thermal loads on the current response of the medium.

Another feature of the model is the fact that system (3.2) does not contain the complete interaction forces but only their dissipative parts $\mathbf{b}_\alpha^{\text{dis}}$. This is due to the presence of the equilibrium component (2.9), which has a universal character, independent of the properties of the fluids.

The temperature gradient and the relative velocities, being "natural" for the heat flux and interaction forces of the continua, by virtue of the "equally present" rule[12] are also included in the number of arguments of all the governing functions. The condition of thermodynamic compatibility further leads to partial splitting – the potentials, stress and entropy of the fluids and the skeleton do not depend on the temperature gradient and relative velocities.

It is assumed that, in the initial state, the temperature and the stress of each continuum are constant in space. The material is assumed to be homogeneous, and hence there are no radius vectors $\mathbf{X}_A$ in a number of the parameters of state. The dependence of the governing components on the relative velocities $\mathbf{w}_\alpha$, and not on the mass velocities $\boldsymbol{v}_A$ themselves, is an obvious consequence of the fact that the reference frame is chosen arbitrarily.

The porous medium, saturated with two fluids, comprises three continua with their initial states. Hence, the symmetry of such a medium is characterized by a set of three groups $g = \{g_s, g_\alpha\}$, comprising the gradients of the transformations of the initial configurations, the governing equations remaining unchanged. These groups depend on the configuration κ_A, and their change on transferring to another initial state is described by Noll's rule.[12]

It is assumed, in the model considered, that the skeleton is a solid material. This means that an undistorted configuration of the skeleton κ_0 exists so that for any configurations of the fluids $\kappa_\alpha(t)$ considered the governing equations possess a group of symmetry $g = \{g_0, u, u\}$. Here $g_0 \in O$ is a subgroup of the characteristic orthogonal group corresponding to the specified type of anisotropy of the skeleton and u is the specific unimodular group of second-rank tensors with a determinant equal to +1.

Another general principle, on which further consideration rests, is that the governing equations are independent of the choice of the reference frame.[12] According to this, relations (3.2) and (3.3) should not change when the reference frame is changed, specified by the transformation

$$\mathbf{x}^* = \mathbf{Q}(t)\cdot(\mathbf{x} - \mathbf{x}_0) + \mathbf{x}_1(t), \quad t^* = t - a, \quad \mathbf{x}_1(-\infty) = \mathbf{x}_0, \quad \mathbf{Q}(-\infty) = \mathbf{I} \tag{3.4}$$

where $\mathbf{Q}(t)$ is the orthogonal tensor of rotation, $\mathbf{x}_1(t) - \mathbf{x}_0$ is the vector of parallel translation, $\mathbf{x}_0 = \text{const}$ is the point about which rotation occurs, and $a = \text{const}$ is the time shift. We will assume, for simplicity, that, at the initial instant of time ($t \to -\infty$), the reference frames coincide. The invariance of the mappings (3.2) and (3.3) with respect to the transformations (3.4) denotes that, in all the reference frames, the state is related to the reaction by one and the same functions $\boldsymbol{\gamma}^+$, $\boldsymbol{\Omega}^+$.

Since, when the reference frame is changed $d\mathbf{x}^* = \mathbf{Q}(t)\cdot d\mathbf{x}$, $d\mathbf{X}_A^* = d\mathbf{X}_A$, we obtain from the definitions of the distortion, the temperature gradient, the relative velocities and the uniqueness of the polar expansions (2.2)

$$\mathbf{G}_A^* = \mathbf{G}_A\cdot\mathbf{Q}^T, \quad \mathbf{V}_A^* = \mathbf{V}_A, \quad \mathbf{R}_A^* = \mathbf{R}_A\cdot\mathbf{Q}^T, \quad \boldsymbol{\gamma}^* = \mathbf{Q}\cdot\boldsymbol{\gamma}, \quad \mathbf{w}_\alpha^* = \mathbf{Q}\cdot\mathbf{w}_\alpha \tag{3.5}$$

It is also assumed that

$$\psi_A^* = \psi_A, \quad \eta_A^* = \eta_A, \quad \theta^* = \theta, \quad \phi_\alpha^* = \phi_\alpha, \quad \mathbf{q}^* = \mathbf{Q}\cdot\mathbf{q}$$
$$\xi^* = \xi, \quad \mathbf{b}_\alpha^{\text{int}*} = \mathbf{Q}\cdot\mathbf{b}_\alpha^{\text{int}}, \quad \boldsymbol{\sigma}_A^* = \mathbf{Q}\cdot\boldsymbol{\sigma}_A\cdot\mathbf{Q}^T \tag{3.6}$$

The third general principle, on which the further investigation rests, is the principle of thermodynamic compatibility[12] (the non-negativity of the uncompensated heat[13]), which requires that inequality (2.12) for the entropy production rate should be satisfied for any history of states. To satisfy the principles of thermodynamic compatibility and independence of the choice of the reference frame, it is necessary and sufficient that the governing equations of the medium considered, with any type of symmetry of the compressed solid skeleton, should have the form

$$\psi_\alpha = \psi_\alpha^\times(\rho_\alpha, \theta), \quad \mathbf{T}_\alpha = -\phi_\alpha p_\alpha \mathbf{I}, \quad p_\alpha = \rho_\alpha^2 \frac{\partial \psi_\alpha^\times}{\partial \rho_\alpha}, \quad \eta_\alpha = -\frac{\partial \psi_\alpha^\times}{\partial \theta} \tag{3.7}$$

$$\psi_s = F(\mathbf{V}, \phi_\beta, \theta, \xi), \quad \mathbf{T}_s = -r_s \frac{\partial F}{\partial \mathbf{G}_s^T} \cdot \mathbf{G}_s, \quad p_\alpha = r_s \frac{\partial F}{\partial \phi_\alpha}, \quad \eta_s = -\frac{\partial F}{\partial \theta} \tag{3.8}$$

$$\Pi_\alpha(\mathbf{V}, \phi_\beta, \rho_\beta, \theta, \xi) \equiv \rho_\alpha^2 \frac{\partial \psi_\alpha(\rho_\alpha, \theta)}{\partial \rho_\alpha} - r_s \frac{\partial F(\mathbf{V}, \phi_\beta, \theta, \xi)}{\partial \phi_\alpha} = 0, \quad \alpha, \beta = 1, 2 \tag{3.9}$$

$$\begin{gathered}\mathbf{q} = \mathbf{R}^T \cdot \mathbf{q}^\times(\mathbf{V}, \phi_\beta, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \quad \mathbf{b}_\alpha^{\text{dis}} = \mathbf{R}^T \cdot \mathbf{b}_\alpha^\times(\mathbf{V}, \phi_\beta, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R) \\ d_s\xi/dt = \Omega^\times(\mathbf{V}, \phi_\beta, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R), \quad \boldsymbol{\gamma}^R \equiv \mathbf{R} \cdot \boldsymbol{\gamma}, \quad \mathbf{w}_\beta^R \equiv \mathbf{R} \cdot \mathbf{w}_\beta\end{gathered} \tag{3.10}$$

$$\delta_c + \delta_f + \delta_T \geq 0, \quad \delta_c = -r_s \frac{\partial F}{\partial \xi} \Omega^\times, \quad \delta_f = \sum_\alpha \mathbf{b}_\alpha^\times \cdot \mathbf{w}_\alpha^R, \quad \delta_T = \theta^{-1} \mathbf{q}^\times \boldsymbol{\gamma}^R \tag{3.11}$$

Here and henceforth $\mathbf{V} = \mathbf{V}_s$ and $\mathbf{R} = \mathbf{R}_s$ are the tension and rotation tensors of an element of the skeleton, and ρ_α and p_α are the true density and pore pressure of the fluid α. The equalities $\Pi_\alpha = 0$ are a system of equations relating the true densities of the fluids with the state of the skeleton. The quantity δ_c is the dissipation of the capillary forces.

It can be seen from formulae (3.7) that the free energy of the fluid α is a function of the true density ρ_α and the temperature of the medium, and is independent of the distortion of the skeleton, the density of the other fluid, the temperature gradient, the relative velocities and the non-equilibrium parameter. The average stress tensor of each fluid is spherical, i.e., the stress deviator is identically equal to zero. This result is directly related to the fact that in the definition of the state of the medium assumed, there are no rates of distortion of the fluids. The pore pressure and the entropy of the fluid are determined by the partial derivatives of its free energy, and hence they are also independent of the temperature gradient and the relative velocities.

The potential of the skeleton depends on its deformation, the temperature, the volume fractions and the non-equilibrium parameter. The average stress tensor and the entropy of the skeleton are determined by its potential and, consequently, are independent of the temperature gradient and the relative velocities. The pore pressures of the fluids are also expressed in terms of partial derivatives of the skeleton potential, which leads to relations between the parameters of state of the skeleton and the true densities of the fluids. This relation is expressed by the system of two equations (3.9), which is formulated using the free energies of the components of the medium, whence it follows, in particular, that even for a fixed deformation, temperature and density of the fluids the volume fractions vary as a consequence of the change in the non-equilibrium parameter.

Formulae (3.10) specify the structure of the expressions for the heat flux, the dissipative interaction forces and the kinetic equation. In general, these functions depend on the temperature gradient and the relative velocities, i.e., the relationship between the heat transfer, the mass and the non-equilibrium nature of the process.

Inequality (3.11) shows that, for any state of the material considered, the total dissipation is equal to the sum of the seepage dissipation, the dissipation by the capillary forces and the thermal dissipation, and, in general, the sum of all the forms of dissipation is non-negative.

The relations obtained hold for arbitrary changes of state. These results are not an assumption but a consequence, proved using the general hypotheses formulated.

The proof of relations (3.7)–(3.11) is quite lengthy and largely repeats the discussions presented in Refs. 8 and 10 for the case of a porous medium saturated with a single fluid. For brevity it is therefore omitted.

The non-uniqueness of the general form. The form of Eqs (3.7)–(3.11), necessary and sufficient to satisfy the principles of objectiveness and thermodynamic compatibility, is not unique. If we change from the variables ϕ_α to the variables (ϕ, S), where $\phi = \phi_1 + \phi_2$ is the porosity and $S = \phi_1/\phi$ is the saturation by the first fluid, then, relations (3.8) in these variables can be written in the form

$$\psi_s = \Phi(\mathbf{V}, \phi, S, \theta, \xi), \; \mathbf{T}_s = -r_s \frac{\partial \Phi}{\partial \mathbf{G}_s^T} \cdot \mathbf{G}_s, \; p = r_s \frac{\partial \Phi}{\partial \phi}, \; \phi p_{\text{cap}} = -r_s \frac{\partial \Phi}{\partial S}, \; \eta_s = -\frac{\partial \Phi}{\partial \theta} \tag{3.12}$$

Instead of the pore pressure p_α, associated with the volume fractions of the fluids ϕ_α, the mean pore and capillary pressure occur here

$$p = Sp_1 + (1 - S)p_2, \quad p_{\text{cap}} = p_2 - p_1 \tag{3.13}$$

In fact, differentiating the free energy of the skeleton as a complex function, we obtain, taking into account the definitions of porosity and saturation,

$$\sum_\alpha \frac{\partial \Phi(\phi(\phi_\gamma), S(\phi_\gamma))}{\partial \phi_\alpha} d\phi_\alpha = \frac{\partial \Phi}{\partial \phi} d\phi + \frac{\partial \Phi}{\partial S} dS$$

where, for brevity, we have omitted the arguments $\mathbf{V}$, θ, ξ of the potential Φ. On the other hand

$$p_1 d\phi_1 + p_2 d\phi_2 = p d\phi - \phi p_{\text{cap}} dS$$

Hence relations (3.12) and (3.13) follow.

Another equivalent form of the equations of a porous medium, saturated with two immiscible fluids, has the form

$$
\begin{aligned}
&\Psi = \Psi(\mathbf{E}, p, S, \theta, \xi)\\
&\mathbf{T} = r_s\mathbf{G}_s^{-1}\cdot\frac{\partial\Psi}{\partial\mathbf{E}}\cdot\mathbf{G}_s^{-1T},\quad \phi = -r_s\frac{\partial\Psi}{\partial p},\quad \phi p_{\text{cap}} = -r_s\frac{\partial\Psi}{\partial S},\quad \eta_s = -\frac{\partial\Psi}{\partial\theta}\\
&\rho_1^2\frac{\partial\psi_1(\rho_1,\theta)}{\partial\rho_1} = p-(1-S)\frac{\partial\Psi/\partial S}{\partial\Psi/\partial p},\quad \rho_2^2\frac{\partial\psi_2(\rho_2,\theta)}{\partial\rho_2} = p+S\frac{\partial\Psi/\partial S}{\partial\Psi/\partial p}\\
&\mathbf{q} = \mathbf{R}^T\cdot\tilde{\mathbf{q}}(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R),\quad \mathbf{b}_\alpha^{\text{dis}} = \mathbf{R}^T\cdot\tilde{\mathbf{b}}_\alpha(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R)\\
&\frac{d_s\xi}{dt} = \Omega(\mathbf{E}, p, S, \theta, \xi, \boldsymbol{\gamma}^R, \mathbf{w}_\beta^R),\quad \delta_c = -r_s\frac{\partial\Psi}{\partial\xi}\Omega,\quad \delta_c+\delta_f+\delta_T\ge 0
\end{aligned}
\tag{3.14}
$$

where $\mathbf{T} = \mathbf{T}_s - \phi p\mathbf{I}$ is the total stress tensor, $\mathbf{E} = (\mathbf{V}^{-2} - \mathbf{I})/2$ is the symmetrical Lagrange finite-strain tensor, while the potential ψ is the solution of the equation

$$\Psi - p\partial\Psi/\partial p = \Phi(\mathbf{G}_s, \phi, S, \theta, \xi) \tag{3.15}$$

For the proof we note that

$$
\begin{aligned}
d\Phi(\mathbf{G}_s, \phi, S, \theta, \xi) &= -r_s^{-1}(\mathbf{T}_s\cdot\mathbf{G}_s^{-1}) : d\mathbf{G}_s^T + r_s^{-1}pd\phi -\\
&- r_s^{-1}\phi p_{\text{cap}}dS - \eta_s d\theta + (\partial\Phi/\partial\xi)d\xi
\end{aligned}
$$

Adding to both sides of this equality the quantity

$$-d(r_s^{-1}p\phi) = \phi p r_s^{-1}\mathbf{G}_s^{-1} : d\mathbf{G}_s^T - r_s^{-1}pd\phi - r_s^{-1}\phi dp$$

obtained taking the formula $dr_s = r_s\mathbf{G}_s^{-1} : d\mathbf{G}_s^T$ into account, and using the relations

$$\left.\frac{\partial\Phi(\phi(p,\xi),\xi)}{\partial\xi}\right|_p = \left.\frac{\partial\Phi}{\partial\xi}\right|_\phi + \left.\frac{\partial\Phi}{\partial\phi}\frac{\partial\phi}{\partial\xi}\right|_p = \left.\frac{\partial\Phi}{\partial\xi}\right|_\phi + r_s^{-1}p\left.\frac{\partial\phi}{\partial\xi}\right|_p = \left.\frac{\partial\Phi}{\partial\xi}\right|_\phi + \left.\frac{\partial(r_s^{-1}p\phi)}{\partial\xi}\right|_p$$

in which, for brevity, we have omitted the arguments $(\mathbf{G}_s, S, \theta)$, we arrive at the identity

$$d(\Phi - r_s^{-1}p\phi) = -r_s^{-1}\mathbf{T}\cdot\mathbf{G}_s^{-1} : d\mathbf{G}_s^T - \eta_s d\theta - r_s^{-1}\phi dp - r_s^{-1}\phi p_{\text{cap}}dS + [\partial(\Phi - r_s^{-1}p\phi)/\partial\xi]d\xi$$

Hence relations (3.14) and (3.15) follow.

For a porous medium with an incompressible skeleton, the true density of which $\rho_s = \text{const}$, it follows from the formula $\phi_1 + \phi_2 = \phi$ and the law of conservation of mass (2.4) that $1 - \phi = (1 - \phi_0)J_s$, where ϕ and ϕ_0 are the current and initial values of the porosity and $J_s = \det\mathbf{G}_s$. This indicates that the current porosity is defined by the determinant of the distortion of the skeleton. Consequently, the governing equations of this medium contain only one independent relation for the volume fractions. This means that in a number of state parameters, instead of the pair (p, S) there is only the argument S,[10] and instead of the total stress tensor there is the effective stress tensor

$$\mathbf{T}^{\text{eff}} \equiv \mathbf{T} + p\mathbf{I} = r_s\mathbf{G}_s^{-1}\cdot\frac{\partial\Psi}{\partial\mathbf{E}}\cdot\mathbf{G}_s^{-1T},\quad p = Sp_1 + (1-S)p_2$$

The above definition of the effective stress tensor generalizes the idea of the effective stress tensor[14] to the case of a porous medium saturated with two immiscible fluids. The reason for the occurrence of effective stresses is the relation between the deformations of the skeleton and the porosity of the medium. Hence, unlike system (3.8) only one equation connecting the true densities of the fluids with the state of the skeleton follows from relation (4.2)

$$\rho_2^2\partial\psi_2(\rho_2,\theta)/\partial\rho_2 - \rho_1^2\partial\psi_1(\rho_1,\theta)/\partial\rho_1 = -\phi^{-1}r_s\partial\Psi(\mathbf{E}, S, \theta, \xi)/\partial S$$

A special case, based on the two assumptions will be investigated below.

First, the rate of change of the non-equilibrium parameter is proportional to the derivative of the potential of the skeleton with respect to this parameter

$$\frac{d_s\xi}{dt} = -\frac{r_s}{\tau(\lambda)}\frac{\partial\Psi(\lambda)}{\partial\xi},\quad \lambda \equiv (\mathbf{G}_s, p, S, \theta, \xi) \tag{3.16}$$

where $\tau(\lambda) > 0$ is a positive function, which depends, in general, on the state parameters and the viscosity of the fluids. As will be seen below, this quantity plays the role of the relaxation time of the capillary pressure as the non-equilibrium parameter tends to its equilibrium value. The order of this quantity can easily be found from an experiment (or calculation) of the time taken for the height of a column of fluid to become established in a gravitational field for an individual straight capillary of specified radius. Estimates of this parameter are known[5] for an element of a porous medium. For a material with kinetics (3.16) the dissipation of the capillary forces

$$\delta_c(\lambda) = -r_s\Omega\partial\Psi/\partial\xi = \tau^{-1}(r_s\partial\Psi/\partial\xi)^2 \ge 0$$

Second, it is assumed that the seepage dissipation and the thermal dissipation are positive in the non-equilibrium process ($w_\alpha \neq 0, \boldsymbol{\gamma} \neq 0$)

$$\delta_f = \sum \mathbf{b}_\alpha^{\text{dis}} \cdot \mathbf{w}_\alpha > 0, \quad \delta_T = \theta^{-1} \mathbf{q} \cdot \boldsymbol{\gamma} > 0 \tag{3.17}$$

Suppose $\Lambda_0 = (\lambda, 0, 0)$ is a quasi-equilibrium state, for which there are no temperature gradient and no relative velocities. The rate of change of the equilibrium parameter in this state, generally speaking, is non-zero. The dissipative forces and thermal flux for the medium considered in this state are equal to zero:

$$\mathbf{q}(\lambda, 0, 0) = 0, \quad \mathbf{b}_\alpha^{\text{dis}}(\lambda, 0, 0) = 0 \tag{3.18}$$

The symmetrical equilibrium thermal conductivity and resistance tensors

$$\mathbf{k}(\lambda) = \partial\mathbf{q}/\partial\boldsymbol{\gamma}|_0, \quad \mathbf{Y}_{\alpha\beta}(\lambda) = 1/2(\partial\mathbf{b}_\alpha^{\text{dis}}/\partial\mathbf{w}_\beta + \partial\mathbf{b}_\beta^{\text{dis}}/\partial\mathbf{w}_\alpha)\big|_0 \tag{3.19}$$

possess the properties of "positive definiteness"

$$\mathbf{k} : (\mathbf{a} \otimes \mathbf{a}) > 0, \quad \sum\sum \mathbf{Y}_{\alpha\beta} : (\mathbf{b}_\alpha \otimes \mathbf{b}_\beta) > 0, \quad \mathbf{a} \neq 0, \quad \mathbf{b}_\alpha \neq 0 \tag{3.20}$$

In fact, in the quasi-equilibrium state the seepage dissipation and the thermal dissipation are equal to zero, i.e., they have a minimum. The necessary condition for an extremum

$$\partial\delta_f/\partial\mathbf{w}_\delta = \mathbf{b}_\beta^{\text{dis}} + \sum \mathbf{w}_\alpha \cdot \partial\mathbf{b}_\alpha^{\text{dis}}/\partial\mathbf{w}_\beta = 0, \quad \theta\partial\delta_T/\partial\boldsymbol{\gamma} = \mathbf{q} + \boldsymbol{\gamma} \cdot \partial\mathbf{q}/\partial\boldsymbol{\gamma} = 0$$

leads to equalities (3.18). The second differentials at the point $\boldsymbol{\gamma} = \mathbf{w}_\alpha = 0$ have the form

$$d^2\delta_T = \theta^{-1}\mathbf{k} : (d\boldsymbol{\gamma} \otimes d\boldsymbol{\gamma}), \quad d^2\delta_f = \sum\sum \mathbf{Y}_{\alpha\beta} : (d\mathbf{w}_\alpha \otimes d\mathbf{w}_\beta)$$

Properties (3.2) follow from the positivity of these expressions when $d\boldsymbol{\gamma} \neq 0$, $d\mathbf{w}_\alpha \neq 0$.

For smooth motions of a porous medium with kinetic non-equilibrium (3.16), the entropy production rate at the point ($\mathbf{x}$,t) is given by the equation

$$\theta \sum_A r_A d_A \eta_A/dt = \delta_c + \delta_f + \nabla \cdot \mathbf{q} + rQ \tag{3.21}$$

In fact, it follows from the energy equation (2.10), taking (2.13) into account, that

$$\sum_\Lambda r_\Lambda \left(\frac{d_\Lambda \psi_\Lambda}{dt} + \eta_\Lambda \frac{d_\Lambda \theta}{dt} + \theta \frac{d_\Lambda \eta_\Lambda}{dt} \right) = -(\mathbf{T}_s \cdot \mathbf{G}_s^{-1}) : \frac{d_s \mathbf{G}_s^T}{dt} + \sum_\alpha p_\alpha \frac{d_s \phi_\alpha}{dt} + \sum_\alpha \frac{p_\alpha \phi_\alpha}{\rho_\alpha} \frac{d_\alpha \rho_\alpha}{dt} + \delta_f + \nabla \cdot \mathbf{q} + \rho r$$

Changing from the energy of the skeleton to the potential Ψ and taking relations (3.7) and (3.14) into account, we arrive at relation (3.21). It can be seen from Eq. (3.21) that the adiabatic process ($\mathbf{q} = 0$, Q = 0) for the porous body considered is not, in general, an isoentropic process (the Joule–Thomson effect). Unlike an equilibrium porous medium for which the entropy production rate is determined by the seepage dissipation, in the material being considered here the entropy production rate also depends on the dissipation by the capillary forces.

4. The quasi-linear approximation

Suppose the displacement of the particles of the skeleton $\mathbf{u} = \mathbf{x} - \mathbf{X}_S$ is small compared with the characteristic dimension of the body L, so that $\mathbf{u}/L = O(\delta)$, $\delta \ll 1$. The gradient of the displacement $\mathbf{H} \equiv \nabla \otimes \mathbf{u}$ and the mass velocities are also assumed to be small. The same assumptions hold for changes in the temperature and the mean threshold pressure. Then, apart from terms $O(\delta^2)$, the distortion of the skeleton $\mathbf{G}_S = \mathbf{I} - \mathbf{H}^T$, and the symmetrical positive-definite tensors $\mathbf{V}_S$, $\mathbf{U}_S$ and the orthogonal tensor $\mathbf{R}_S$ in expansion (2.2) are such that

$$\mathbf{U}_s = \mathbf{V}_s = \mathbf{I} - \mathbf{e}, \quad \mathbf{R}_s = \mathbf{I} + \boldsymbol{\omega}$$

where $\mathbf{e} = ((\nabla \otimes \mathbf{u})^T + \nabla \otimes \mathbf{u})/2$ is the small-strain tensor and $\boldsymbol{\omega} = ((\nabla \otimes \mathbf{u})^T - \nabla \otimes \mathbf{u})/2$ is the antisymmetric small-rotation tensor of an element of the skeleton.

In the case of a weakly compressed fluid the free energy in the neighbourhood of the initial state (ρ_α^0, θ_0) can be represented in the form of a quadratic expansion

$$\rho_\alpha^0 \psi_\alpha(\Delta\rho_\alpha, \vartheta) = \rho_\alpha^0 \psi_\alpha^0 - \rho_\alpha^0 \eta_\alpha^0 \vartheta + p_\alpha^0 I_\alpha + K_\alpha I_\alpha^2/2 + \alpha_\alpha \vartheta I_\alpha - \rho_\alpha^0 c_\alpha \theta_0^{-1} \vartheta^2/2$$

where (ψ_a^0, p_α^0) and η_α^0 are the energy, pressure and entropy of the fluid in the initial state, $\vartheta = \theta - \theta_0$ is the change in temperature, $I_\alpha = \Delta\rho_\alpha/\rho_\alpha^0$ is the relative change in the density, K_α is the bulk compression modulus, α_α is the coefficient of thermal expansion and c_α is the heat capacity at constant volume. Hence we obtain the following expressions for the pore pressure and entropy

$$p_\alpha = p_\alpha^0 + K_\alpha I_\alpha + \alpha_\alpha \vartheta, \quad \eta_\alpha = \eta_\alpha^0 - \alpha_\alpha I_\alpha/\rho_\alpha^0 + c_\alpha \vartheta/\theta_0$$

The state of the compressed skeleton, saturated with two fluids, is specified by the set of quantities ($\mathbf{E}$, p, θ, S, ξ), while the reaction is characterized by the quantities

$$\Psi = \Psi(\mathbf{E}, p, \theta, S, \xi), \quad \mathbf{T} = r_s \mathbf{G}_s^{-1} \cdot (\partial\Psi/\partial\mathbf{E}) \cdot \mathbf{G}_s^{-1T}, \quad \eta_s = -\partial\Psi/\partial\theta,$$
$$\phi = -Z_s \partial\Psi/\partial p, \quad \phi p_{\mathrm{cap}} = -r_s \partial\Psi/\partial S, \quad \dot{\xi} = -\tau^{-1} r_s \partial\Psi/\partial\xi \tag{4.1}$$

The dot denotes a partial derivative with respect to time, which, apart from terms $O(\delta^2)$, is identical with the derivative d_s/dt along the trajectory of a skeleton particle.

Suppose that, in the initial state, the total stress $\mathbf{T}_0$, the mean pore pressure p_0, the temperature θ_0, the saturation S_0 and the parameter ξ are specified. The external action causes small displacements of the points of the skeleton, and a change in the mean pore pressure and temperature. The small displacements cause a distortion of the skeleton, close to the unit tensor, and a strain $\mathbf{E} = \mathbf{e} + O(\delta^2)$. The change in the saturation and in the non-equilibrium parameter is, generally speaking, finite. In order to reflect this feature of the medium, we will introduce a so-called reference state, in which $\mathbf{E} = 0$, $p = p_0$, $\theta = \theta_0$. The reference state differs from the initial state in that the effective saturation and non-equilibrium parameter in it are equal to their current values. We will assume that the potential $\Psi(\mathbf{E}, p, \theta, S, \xi)$ is a smooth function of the variables ($\mathbf{E}$, p, θ) in the neighbourhood of the reference state. Then, expanding the potential in a Taylor series, we obtain

$$r_s^0 \Psi(\mathbf{E}, p, \vartheta, S, \xi) = r_s^0 \Psi_* + \mathbf{T}_* : \mathbf{E} - \phi_* \Delta p - r_s^0 \eta_* \vartheta + \mathbf{e} : \mathbf{L} : \mathbf{e}/2 -$$
$$- N^{-1}(\Delta p)^2/2 - \mathbf{B} : \mathbf{e}\Delta p - \mathbf{A}_\theta : \mathbf{e}\vartheta + \alpha_\phi \vartheta \Delta p - \theta_0^{-1} r_s^0 c_s \vartheta^2/2 + O(\delta^3) \tag{4.2}$$

The subscript asterisk indicates that the quantities are taken in the reference state. Hence, the first term and the coefficients of the expansion are functions of (S,ξ). If an element of the porous medium in the reference state is completely saturated by the first, more wetting fluid ($S = 1$), the parameter ξ and the potential Ψ_* in this state are equal to zero. The values $\xi > 0$ and $\Psi_*(S, \xi) \geq 0$ correspond to $0 < S < 1$.

The following equalities hold for the first derivatives of the potential in the reference state

$$\partial\Psi/\partial\mathbf{E}|_* = \mathbf{T}_*(S, \xi)/r_s^0, \quad \partial\Psi/\partial p|_* = -\phi_*(S, \xi)/r_s^0, \quad \partial\Psi/\partial\theta|_* = -\eta_s^*(S, \xi)$$

The legitimacy of these assertions follows from formulae (4.1).

The second derivatives of the potential represent the heat capacity, the moduli of elasticity and the expansion coefficients of the porous medium. Suppose $c_s(S, \xi) = -\theta_0 \partial^2\Psi/\partial\theta^2|_*$ is the heat capacity of the skeleton and $\mathrm{L}(S, \xi) = r_s^0 \partial^2\Psi/\partial\mathbf{E} \otimes \partial\mathbf{E}|_*$ is a fourth-rank moduli of elasticity tensor of the skeleton material. The symmetrical tensor of the thermoelastic coefficients $\mathbf{A}_\theta(S, \xi) = -r_s^0 \partial^2\Psi/\partial\theta\partial\mathbf{E}|_*$ represents the change in the stress when there is a change in temperature, and also the increment of the entropy of the skeleton when its strain changes. The Biot coefficients tensor $\mathbf{B}(S, \xi) = -r_s^0 \partial^2\Psi/\partial p\partial\mathbf{E}|_*$ relates the change in the porosity ϕ to the strain increment. On the other hand, this tensor defines the dependence of the stress on the mean pore pressure. The scalar Biot modulus $N^{-1}(S, \xi) = -r_s^0 \partial^2\Psi/\partial p^2|_*$ relates the change in the porosity ϕ to the change in the pore pressure p. The coefficient $\alpha_\phi(S, \xi) = r_s^0 \partial^2\Psi/\partial p\partial\theta|_*$ represents the change in the porosity ϕ when there is a change in temperature, and also relates the change in the entropy to the increment in the mean pore pressure.

In expansion (4.2) the tensor of finite strains $\mathbf{E}$ can be replaced by the small strain tensor $\mathbf{e}$ if the stresses in the reference state are small compared with the other moduli. If these quantities are of the same order, then, when the tensor $\mathbf{E}$ is replaced by the small-strain tensor $\mathbf{e}$ in the term $\mathbf{T}_*$:$\mathbf{E}$, terms of order $O(\delta^2)$, which were retained in expansion (4.2), will be lost.

If we use the formula for the mass density $r_s = r_s^0(1 - \mathbf{e} : \mathbf{I}) + O(\delta^2)$, the relation $\mathbf{G}_s = I - (\nabla \otimes \mathbf{u})^T + O(\delta^2)$ and expansion (4.2), a calculation of the total stress using the formula $\mathbf{T} = r_s \mathbf{G}_s^{-1} \cdot (\partial\Psi/\partial\mathbf{E}) \cdot \mathbf{G}_s^{-1T}$ gives the linear relation

$$\mathbf{T} = \mathbf{T}_* + \boldsymbol{\omega} \cdot \mathbf{T}_* - \mathbf{T}_* \cdot \boldsymbol{\omega} - (\mathbf{I} : \mathbf{e})\mathbf{T}_* + \mathbf{T}_* \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{T}_* + \mathbf{L} : \mathbf{e} - \mathbf{A}_\theta \vartheta - \mathbf{B}\Delta p$$

Hence we see that an increment of the stress tensor $\mathbf{T} - \mathbf{T}_*$ due to a small change in the variables ($\mathbf{e}$, Δp, ϑ), which is equal to zero in the reference state, represents the sum of three terms

$$\boldsymbol{\omega} \cdot \mathbf{T}_* - \mathbf{T}_* \cdot \boldsymbol{\omega}, \quad \mathbf{T}_* \cdot \mathbf{e} + \mathbf{e} \cdot \mathbf{T}_* - (\mathbf{e} : \mathbf{I})\mathbf{T}_*, \quad \mathbf{L} : \mathbf{e} - \mathbf{A}_\theta \vartheta - \mathbf{B}\Delta p$$

In theories of the elasticity of porous media, saturated with a single fluid, the reference state is identical with the initial state. Hence, for a specified displacement, the first and second terms are the same for all elastic materials, and only the third term depends on the properties of the medium. For a porous medium saturated with several immiscible fluids, the situation is different since the stresses $\mathbf{T}_*$ depend on the properties of the material.

Suppose the stresses in the reference state are small compared with the moduli of elasticity. Then, the first two of these terms are of the order of δ^2 and can be neglected. It is this case that is considered below. Then, the total stress, porosity, entropy and capillary pressure are given by the relations

$$\mathbf{T} = \mathbf{T}_*(S, \xi) + \mathbf{L}(S, \xi) : \mathbf{e} - \mathbf{A}_\theta(S, \xi)\vartheta - \mathbf{B}(S, \vartheta)\Delta p$$
$$\phi = \phi_*(S, \xi) + \mathbf{B}(S, \xi) : \mathbf{e} + \Delta p/N(S, \xi) - \alpha_\phi(S, \xi)\vartheta \tag{4.3}$$
$$r_s^0 \eta_s = r_s^0 \eta_*(S, \xi) + \mathbf{A}_\infty(S, \xi) : \mathbf{e} - \alpha_\phi(S, \xi)\Delta p + r_s^0 c_s(S, \xi)\vartheta/\theta_0$$

$$-\phi p_{\mathrm{cap}} = r_s \partial\Psi_*/\partial S + \mathbf{e} : \partial\mathbf{T}_*/\partial S - \Delta p \partial\phi_*/\partial S - r_s^0 \vartheta \partial\eta_*/\partial S \tag{4.4}$$

The equation of kinetics (3.16) in this case takes the form

$$\frac{\partial\xi}{\partial t} = -\frac{1}{\tau}\left[r_s\frac{\partial\Psi_*}{\partial\xi} + \frac{\partial\mathbf{T}_*}{\partial\xi} : \mathbf{e} - \frac{\partial\phi_*}{\partial\xi}\Delta p - r_s\frac{\partial\eta_*}{\partial\xi}\vartheta\right] \tag{4.5}$$

The derivative $\partial\Psi/\partial\xi$ can be considered as the sum of two terms – the energy absorption rate, expended in increasing the contact surface of a non-wetting fluid with the skeleton, and the energy release rate when the parameter ξ increases, i.e., when approaching the equilibrium state. The condition $-\partial\Psi/\partial\xi > 0$ denotes that the approach to equilibrium state is accompanied by energy dissipation, i.e., the energy released is greater than the energy absorbed.

In the case when the relaxation time τ is small compared with the characteristic time of the process, an equilibrium change in the parameter ξ occurs. It follows from kinetic equation (4.5) that as $\tau \to 0$ the rate ξ is bounded, if

$$r_s\frac{\partial\Psi_*(S,\xi)}{\partial\xi} + \frac{\partial\mathbf{T}_*(S,\xi)}{\partial\xi} : \mathbf{e} - \frac{\partial\phi_*(S,\xi)}{\partial\xi}\Delta p - r_s\frac{\partial\eta_*(S,\xi)}{\partial\xi}\vartheta = 0 \tag{4.6}$$

This denotes that, in this process, the non-equilibrium parameter is a function of the current values of the strain, the pore pressure, the temperature and the saturation.

The governing equations for the heat flux and the interaction forces. Suppose the heat flux and the dissipative interaction forces are smooth functions in the neighbourhood of the reference state. We then obtain the following relations in the linear approximation

$$\mathbf{q} = \mathbf{k}(S,\xi)\cdot\boldsymbol{\gamma}, \quad \mathbf{b}_\alpha^{\text{dis}} = \sum\mathbf{Y}_{\alpha\beta}(S,\xi)\cdot\mathbf{w}_\beta \tag{4.7}$$

which contain symmetrical equilibrium thermal conductivity and resistance tensors

$$\mathbf{k}(S,\xi) = \left.\frac{\partial\mathbf{q}}{\partial\boldsymbol{\gamma}}\right|_{\boldsymbol{\gamma}=0}, \quad \mathbf{Y}_{\alpha\beta}(S,\xi) = \left.\frac{1}{2}\left(\frac{\partial\mathbf{b}_\alpha^{\text{dis}}}{\partial\mathbf{w}_\beta} + \frac{\partial\mathbf{b}_\beta^{\text{dis}}}{\partial\mathbf{w}_\alpha}\right)\right|_{\mathbf{w}_\alpha=0}$$

In expansions (4.7) we have taken into account the fact that, at the point $\boldsymbol{\gamma} = 0$, the heat flux is equal to zero for all $(\mathbf{e}, p, \vartheta, \mathbf{w}_\alpha)$, as a result of which the derivatives of the heat flux and dissipative interaction forces arising from the strain, pressure, temperature and velocities are also equal to zero, i.e., the effect of these factors on the heat conduction is small in the linear approximation. Similar assertions also hold for the dissipative force.

In applications, instead of the resistance tensors it is more convenient to use tensors that are the inverse of these, namely, the true permeability tensors

$$\mathbf{K}_{\alpha\beta} = \phi_\alpha\phi_\beta(\mu_\alpha\mu_\beta)^{1/2}\mathbf{Y}_{\alpha\beta}^{-1} \tag{4.8}$$

where ϕ_α are volume fractions and μ_α are the viscosities of the fluids. By virtue of the second formula of (4.3) the porosity $\phi = \phi(\mathbf{e}, \Delta p, \vartheta, S, \xi)$, while the volume fractions of the fluids $\phi_1 = \phi S$, $\phi_2 = \phi(1 - S)$. Hence, the expressions for the true permeability tensors have the form

$$\mathbf{K}_{11} = \phi^2 S^2\mu_1\mathbf{Y}_{11}^{-1}(S,\xi), \quad \mathbf{K}_{22} = \phi^2(1-S)^2\mu_2\mathbf{Y}_{22}^{-1}(S,\xi), \quad \mathbf{K}_{12} = \phi^2 S(1-S)(\mu_1\mu_2)^{1/2}\mathbf{Y}_{12}^{-1}(S,\xi)$$

so that even for constant resistance tensors the most important feature of the combined seepage of two immiscible fluids, namely, the vanishing of the permeability of one fluid when the medium is totally saturated with the other fluid, is taken into account. To describe the capillary sticking (the residual saturation), when the motion of the fluid ceases when the medium is incompletely saturated with the other fluid, we must introduce the relations $Y_{\alpha\beta}^{-1}(S,\xi)$.

Taking representation (2.8) and (2.9) for the interaction forces into account and using relation (4.9), equation of motion (2.7) can be written in the form

$$-\phi_\alpha(\nabla p_\alpha - \rho_\alpha(\mathbf{g} - \mathbf{a}_\alpha)) = \sum\mathbf{Y}_{\alpha\beta}(S,\xi)\mathbf{w}_\beta, \quad \mathbf{a}_\alpha \equiv d_\alpha\boldsymbol{\upsilon}_\alpha/dt$$

Hence, when taking definition (4.8) for the true permeability tensor into account, we must use the generalized Darcy's law for multiphase seepage

$$\mathbf{W}_\alpha = -\sum(\mu_\alpha\mu_\beta)^{-1/2}\mathbf{K}_{\alpha\beta}\cdot(\nabla p_\beta - \rho_\beta(\mathbf{g} - \mathbf{a}_\beta)) \tag{4.9}$$

where $\mathbf{W}_\alpha = \phi_\alpha\mathbf{w}_\infty$ is the partial seepage vector.

We emphasise that the governing relations (4.2)-(4.9) hold for a medium with an arbitrary type of skeleton anisotropy. Further simplification is achieved for a medium with an initially isotropic skeleton. In this case an initial configuration of the skeleton κ_0 exists, the symmetry group in which is identical with the rotation group. The thermodynamic potential is a scalar function of independent invariants of the strain tensor, while the heat-flux vectors and the interaction forces satisfy the isotropy condition.

5. Seepage of fluids in an undeformed skeleton

The seepage of immiscible fluids, saturating the skeleton of a porous medium and differing in density and viscosity, is of considerable interest in the petroleum industry, being the main irrigation technology which enables petroleum extraction to be intensified. However, the process of displacing a viscous fluid by another less viscous fluid is accompanied by the formation of so-called "viscous fingers", which lead to the break through of water into mining boreholes. This phenomenon, known as Saffman–Taylor instability,[15] is easily observed using a Hele-Shaw cell.[16] Hence, it is urgent to try to construct a model which reflects the basic features of the displacement process and which will enable the effect of various factors to be predicted.

The Muskat–Leverett model gives the most complete description of this process. Unlike the traditional description,[5] here the non-equilibrium nature of the process is taken into account as well as the cross terms in Darcy's law related to. The deformation of the skeleton and the change in the temperature is neglected. The fluids, with true densities ρ_α, are assumed to be incompressible. The skeleton is assumed to be homogeneous and isotropic. The porosity ϕ and the permeability k are constant quantities. The viscosities of the fluids $\mu_\alpha = \text{const}$, and to be specific we assume $\mu_1 < \mu_2$. We will use the quasi-static approximation, when the inertial forces can be neglected.

We will take the potential $\Psi_* = (S, \xi)$, which occurs in expressions (4.2), in the form

$$r_s\Psi_*(s,\xi) = \xi^2/2 + Ms^2/2 - \alpha\xi s + \gamma s, \quad s = 1 - S, \tag{5.1}$$

where r_s, α, γ, $M > 0$ are constant quantities, and s is the saturation of the skeleton by the second less-wetting fluid. The surface $y = \Psi_*(s, \xi)$ is an elliptic paraboloid, since, in the variables

$$x = s + \gamma/M_{\text{eq}}, \quad y = \xi - s\alpha, \quad M_{\text{eq}} = M - \alpha^2$$

the function $\Psi_*(x, y)$ has the form

$$2\Psi_*(x,y) = y^2 + M_{\text{eq}}^2 x^2 - \gamma^2/M_{\text{eq}}$$

For the potential (5.1) the capillary pressure, by virtue of relations (4.4), is given by the expression

$$\phi p_{\text{cap}}(s,\xi) = \gamma + Ms - \alpha\xi \tag{5.2}$$

When $S = 1$ the parameter $\xi = 0$, since there is no contact surface of the less wetting fluid with the skeleton. Hence γ is the input capillary pressure for which displacement of the initial more wetting fluid from the porous medium begins and M is the instantaneous modulus of the capillary pressure.

The equation of kinetics (4.5), corresponding to potential (5.1), has the form

$$\dot{\xi} = (\alpha s - \xi)/\tau \tag{5.3}$$

The equilibrium value of the parameter ξ is determined by the closed form relation $\xi_{\text{eq}}(s) = \alpha s$, from which it follows that, in the equilibrium process, expression (5.2) takes the form

$$\phi p_{\text{cap}}(s) = \gamma + M_{\text{eq}} s \tag{5.4}$$

We will consider an example. Suppose the relaxation time $\tau = \text{const}$ and the saturation when $t < 0$ is equal to $s_0 < 1$. The parameter ξ in this state is equal to $\xi_0 = \alpha s_0$, while the capillary pressure is given by Eq. (5.4) with $s = s_0$. At the instant $t = 0$ the value of s changes abruptly to the value $s_1 > s_0$. The solution of the kinetic equation (5.3) with initial value $\xi(0) = \xi_0$ has the form

$$\xi(t) = \alpha(s_1 - \Delta s\exp(-t/\tau)), \quad \Delta s = s_1 - s_0$$

while the capillary pressure depends on the time as follows:

$$\phi p_{\text{cap}}(t) = \gamma + M_{\text{eq}}s_1 + \Delta s\alpha^2\exp(-t/\tau)$$

Hence it can be seen that at the instant $t = +0$ the change in the capillary pressure is determined by the instantaneous modulus, since $\phi\Delta p_{\text{cap}}(+0) = M\Delta s$.

Suppose that, after relaxation of the capillary pressure at saturation $s = s_1$ to a value close to the equilibrium value $\gamma + M_{\text{eq}}s_1$, the saturation at the instant t_0 again changes abruptly to the previous value s_0. In this state the capillary pressure is

$$\phi p_{\text{cap}}(t_1) = \gamma + M_{\text{eq}}s_0 - \Delta s\alpha^2\exp(-t_1/\tau), \quad t_1 = t - t_0$$

When $t_0 = +0$ the change in the non-equilibrium parameter is equal to zero, while the increment of the capillary pressure $\phi\Delta p_{\text{cap}} = -M\Delta s$, i.e., it is also determined by the instantaneous modulus M.

Hence, the maximum width of the hysteresis loop, described around the equilibrium relation (5.4) in the (p_{cap}, s) plane, is proportional to the change in the saturation Δs with coefficient $\alpha^2 = M - M_{\text{eq}}$. The maximum slope of the hysteresis curve is determined by the instantaneous modulus of the capillary pressure M.

The above example shows that it is possible in principle to determine the parameters γ, α, τ and M from experiments on the relaxation of the capillary pressure.

We will now consider the equations of multiphase seepage. Suppose $\mathbf{W}_1 = \phi S v_1$ and $\mathbf{W}_2 = \phi(1-S)v_2$ are the partial-seepage vectors and $\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2$ is the total seepage vector. The total seepage velocity field is solenoidal:

$$\nabla\cdot\mathbf{W} = 0 \tag{5.5}$$

while the saturation satisfies the parabolic equation

$$\frac{\partial\phi S}{\partial t} + \nabla\cdot[B(S,\xi)\mathbf{W} + n(S,\xi)\nabla p_{\text{cap}}(S,\xi) + \mathbf{g}G(S,\xi)] = 0 \tag{5.6}$$

where

$$B = n(m_{22} - m_{21}), \quad n = (m_{11} - 2m_{21} + m_{22})^{-1}, \quad G = kn\sum\sum(m_{1\beta} - m_{2\beta})f_{\beta\lambda}\rho_\lambda/(\mu_\beta\mu_\lambda)^{1/2}$$

$m_{\alpha\beta}$ is the matrix, inverse to the conduction matrix $k_{\alpha\beta} = kf_{\alpha\beta}/(\mu_\alpha\mu_\beta)^{1/2}$, k is the absolute permeability and $f_{\alpha\beta}(S, \xi)$ are the relative phase permeabilities.

To prove relations (5.5) and (5.6) we turn to the generalized Darcy's law (4.9). For an isotropic skeleton this law has the form

$$\mathbf{W}_\alpha = -\sum k_{\alpha\beta}\nabla p_\beta + \mathbf{g}G_\alpha, \quad G_\alpha = k\sum f_{\alpha\beta}\rho_\beta/(\mu_\alpha\mu_\beta)^{1/2}$$

By virtue of property (3.19) the matrix $k_{\alpha\beta}$ is the non-degenerate, and hence it follows that

$$\nabla p_\alpha = \sum m_{\alpha\beta}(\mathbf{g}G_\beta - \mathbf{W}_\beta), \quad \sum m_{\alpha\beta}k_{\beta\gamma} \equiv \delta_{\alpha\gamma} \tag{5.7}$$

Expressing the partial pressures p_α in terms of the mean pore and capillary pressure using the formulae

$$p_1 = p - (1 - S)p_{\text{cap}}, \quad p_2 = p + Sp_{\text{cap}}$$

we convert relations (5.7) to the form

$$\nabla p = \sum m_{1\beta}(\mathbf{g}G_\beta - \mathbf{W}_\beta) + \nabla[(1 - S)p_{\text{cap}}], \quad \nabla p = \sum m_{2\beta}(\mathbf{g}G_\beta - \mathbf{W}_\beta) - \nabla[Sp_{\text{cap}}]$$

Subtracting the second equation from the first, we obtain

$$\sum(m_{1\beta} - m_{2\beta})\mathbf{W}_\beta = \mathbf{g}\sum(m_{1\beta} - m_{2\beta})G_\beta + \nabla p_{\text{cap}}$$

Using the definition of the total seepage vector, we arrive at a system of two linear equations in $\mathbf{W}_\beta$, the solution of which has the form

$$\mathbf{W}_1 = B\mathbf{W} + n\nabla p_{\text{cap}} + \mathbf{g}G, \quad \mathbf{W}_2 = (1 - B)\mathbf{W} - n\nabla p_{\text{cap}} - \mathbf{g}G \tag{5.8}$$

The functions $B(S, \xi)$, $n(S, \xi)$, $G(S, \xi)$ were defined above. The function B plays an important role in seepage theory and is called the generalized Buckley function. It can be seen from formulae (5.8) that, when there is no gravitational force and when the pore pressures are equal this function gives the fraction of the first fluid in the total flow.

We now turn to the continuity equations (2.5), which, for incompressible fluids and an undeformed skeleton, can be written in the form

$$\frac{\partial\phi S}{\partial t} + \nabla\cdot\mathbf{W}_1 = 0, \quad \frac{\partial\phi(1 - S)}{\partial t} + \nabla\cdot\mathbf{W}_2 = 0$$

Adding these relations we obtain Eq. (5.5). Substituting the first relation of (5.8) into the equation of continuity for the first fluid, we arrive at Eq. (5.6).

Equation (5.6) becomes simplified if the relative phase permeabilities and the functions B, n and G depend on a single variable – the effective saturation $\omega = S + \alpha\xi/M$, which, in view of Eq. (5.2) determines the capillary pressure. Another basis for introducing an effective saturation[5] is the large value of the phase permeability of the wetting fluid compared with its equilibrium value when the saturation increases. This is due to the fact that, in the non-equilibrium state, part of the displacing fluid temporarily moves along wider capillary channels than in the steady flow.

Using the kinetic equation (5.3), we obtain

$$\frac{\partial S}{\partial t} = \frac{\partial\omega}{\partial t} - \frac{\alpha}{\tau M}\Lambda(\omega, \xi); \quad \Lambda(\omega, \xi) = \alpha(1 - \omega) - \frac{M_{\text{eq}}}{M}\xi$$

Substituting this expression into Eq. (5.6), we arrive at a divergent equation for the effective saturation

$$\frac{\partial\phi\omega}{\partial t} + \nabla\cdot[B(\omega)\mathbf{W} - A(\omega)\nabla\omega + \mathbf{g}G(\omega)] = \frac{\alpha\phi}{\tau M}\Lambda(\omega, \xi), \quad A(\omega) = \phi^{-1}Mn(\omega)$$

Taking Eq. (5.5) for the rate of complete seepage into account it follows that

$$\frac{\partial\phi\omega}{\partial t} + (B'(\omega)\mathbf{W} + G'(\omega)\mathbf{g})\cdot\nabla\omega - \nabla\cdot(A(\omega)\nabla\omega) = \frac{\alpha\phi}{\tau M}\Lambda(\omega, \xi) \tag{5.9}$$

The closed system of equations (5.3), (5.5) and (5.9) in terms of the unknowns **W**, α, ξ will be called the modified Muskat–Leverett model.

Suppose the functions $f_{\alpha\beta}(\omega)$ are defined by the expressions

$$f_{11}(\omega) = \omega^q, \quad f_{22}(\omega) = (1 - \omega)^q, \quad f_{12}(\omega) = m(f_{11}f_{22})^{1/2}, \quad 0 < m < 1, \quad q > 1 \tag{5.10}$$

A typical form of relations (5.10) is shown in Fig. 1. Graphs of $B(\omega)$, $B'(\omega)$, $A(\omega)$ for values of the parameters $q = 2.5$ and $\mu_1/\mu_2 = 0.2$ are shown in Fig. 2 by the continuous curves for $m = 0.83$ and by the dashed curves for $m = 0$, i.e., in the case when there are no cross terms. It can be seen that these functions differ considerably. The Buckley function B(ω) for the case when $m > 0$ is convex upwards with the exception of a small part at low saturations. The maximum of the function B'(ω) when $m \neq 0$ is shifted into the region of low saturations, and for these parameters is approximately half the maximum when there are no cross terms.

Using the non-equilibrium model we will consider the one-dimensional problem of the displacement of petroleum by water from a thin horizontal layer $0 \leq x \leq x_0$, initially saturated with petroleum. In this case we can neglect the effect of the gravity force. It is assumed that

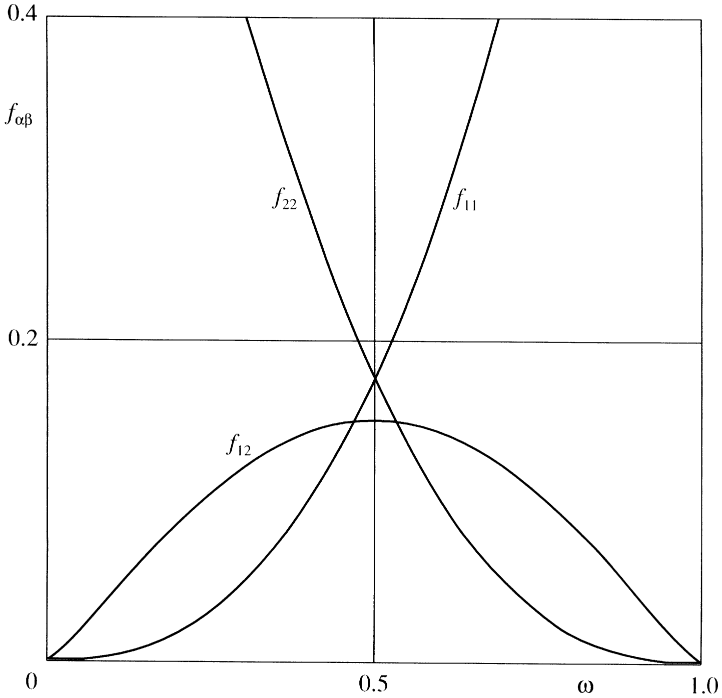

Fig. 1.

a constant flow of water W_0 per unit area of the boundary $x = 0$ is specified. In view of Eq. (5.5) $W(x, t) = W_0$ everywhere. To be specific we will assume that the layer is hydrophilic, and the saturating liquids are water and petroleum. Changing to dimensionless variables

$$\bar{x} = x/x_0, \quad \bar{t} = t/t_0 = tW_0/\phi x_0, \quad \bar{\tau} = \tau/t_0$$

and henceforth omitting the bar over the dimensionless variables and introducing the notation

$$A(\omega) = \varepsilon a(\omega), \quad \varepsilon = Mk_0/(\phi x_0\mu_0 W_0), \quad \eta = t_0/\tau = \phi x_0/(\tau W_0) \tag{5.11}$$

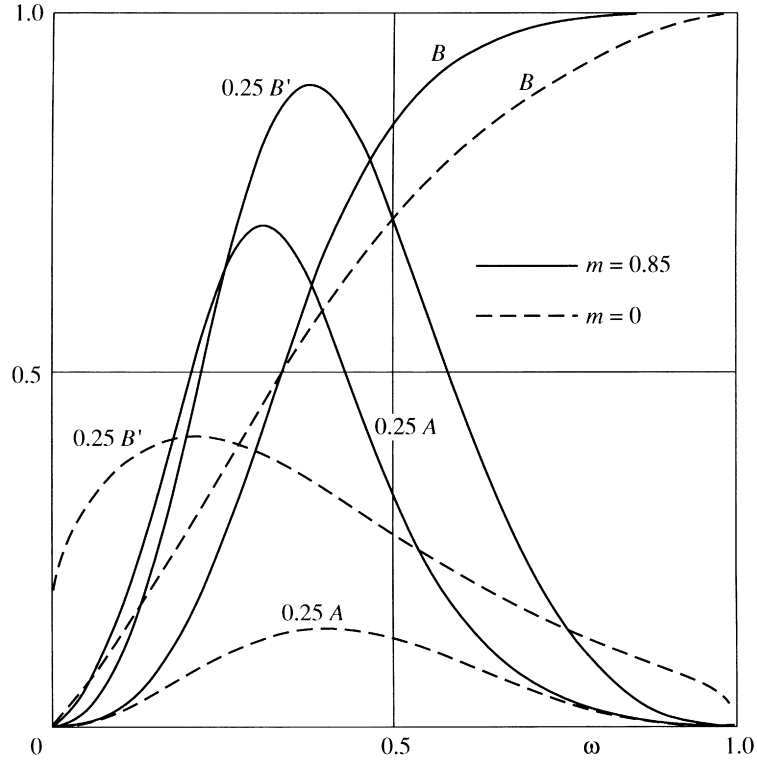

Fig. 2.

Eq. (5.9) for the evolution of the saturation and the kinetic equation (5.3) can be written in the form

$$\tau\left[\frac{\partial\omega}{\partial t}+B'(\omega)\frac{\partial\omega}{\partial x}\right]-\varepsilon\tau\frac{\partial}{\partial x}\left[a(\omega)\frac{\partial\omega}{\partial x}\right] = \frac{\alpha}{M}\Lambda(\omega,\xi),\quad \tau\frac{\partial\xi}{\partial t} = \Lambda(\omega,\xi) \tag{5.12}$$

Suppose the characteristic quantities have the following values: the length of the layer $x_0 = 100$ m, the velocity $v_0 = W_0/\phi = 10^{-3}$ m/s, the viscosity $\mu_0 = 10^{-2}$ Pa s, the porosity $\phi = 10^{-2}$, the permeability $k_0 = 10^{-14}$ m^2, the modulus of the capillary pressure $M = 10^4$ Pa, the relaxation time $\tau = 10^{-3}\ t_0$ and the parameter $\alpha = 0.1$ M. Then the characteristic time $t_0 = \phi x_0/W_0 = 10^5$ s, and the parameters $\varepsilon = 10^{-3}$ and $\tau = 10^2$. Hence, the first equation of (5.12) contains the product of small parameters for the leading derivative ($\varepsilon\tau \ll 1$) and a small parameter for the first derivatives. In the second equation of (5.12) we have a small parameter for the first derivative.

The first equation of (5.12) is a quasi-linear parabolic-type equation for which the initial and boundary conditions in the sections $x = 0$ and $x = 1$ must be specified. We will assign the initial condition, corresponding to equilibrium saturation $S(x, 0) = 0$, in the form

$$\omega(x,0) = \omega_0 \equiv \alpha^2/M,\quad \xi(x,0) = \xi_0 \equiv \alpha,\quad 0 \le x \le 1 \tag{5.13}$$

The boundary condition when $x = 0$, corresponding to $S(0, t) = 1$ and $\xi(0, t) = 0$ has the form

$$\omega(0,t) = 1 \tag{5.14}$$

We will assume that, on the right boundary $x = 1$, the volume content of water is equal to zero, and hence $\xi(1, t) = \xi_0$ and the effect of saturation is

$$\omega(1,t) = \omega_0 \tag{5.15}$$

We will consider the approximate solution of boundary-value problem (5.12)–(5.15) in the form of power expansions. Suppose, to be specific, the parameters ε and τ are of the same order of smallness, i.e., $\varepsilon/\tau = O(1)$. Writing the continuous solution in the form of series

$$\omega = \omega^{(0)}+\omega^{(1)}\varepsilon+\omega^{(2)}\varepsilon^2+\ldots,\quad \xi = \xi^{(0)}+\xi^{(1)}\varepsilon+\xi^{(2)}\varepsilon^2+\ldots$$

and substituting into Eqs (5.12), we obtain, after equating like powers of the small parameter ε, a recurrent system of equations. The quantities $\Lambda(\omega^{(0)}, \xi^{(0)}) = 0$ correspond to the zero approximation. This means that the zero approximation is the equilibrium approximation, for which

$$\xi^{(0)} = (1-\omega^{(0)})\alpha M/M_{\mathrm{eq}},\quad \omega^{(0)} = M_{\mathrm{eq}}M(S^{(0)}+\alpha^2/M_{\mathrm{eq}})$$

Substituting the expression for the effective saturation $\omega^{(0)}$ into the initial equation (5.6), we obtain a hyperbolic equation

$$\frac{\partial\omega^{(0)}}{\partial t}+B'_{\mathrm{eq}}(\omega^{(0)})\frac{\partial\omega^{(0)}}{\partial x} = 0,\quad B_{\mathrm{eq}}(\omega^{(0)}) \equiv B(\omega^{(0)})\frac{M_{\mathrm{eq}}}{M} \tag{5.16}$$

where $B_{\mathrm{eq}}(\omega^{(0)})$ is the Buckley equilibrium function, which differs from the initial Buckley function by the factor $\mathrm{M_{eq}/M}$. The characteristics of Eq. (5.16) are straight lines with slope $dx/d\tau = B'_{\mathrm{eq}}(\omega^{(0)})$. The effective saturation is constant along the characteristics.

The function $\mathrm{B}(\omega)$, represented in Fig. 2 by the continuous curve, is complex upwards everywhere, apart from the part corresponding to small values of the saturation. The equilibrium function $\mathrm{E_{eq}}(\omega)$, which corresponds to the velocities of the characteristics, which decrease as ω increases, possesses the same property. Hence, in the neighbourhood of the boundary $x = 0$, where $\omega(0, t) \sim 1$, the solution is continuous and is defined by the relation $x/t = B'_{\mathrm{eq}}(\omega^{(0)})$. Hence it follows that $\omega^{(0)}(x, t) = z(x/t)$, where z is a function inverse to $B'_{\mathrm{eq}}(\omega^{(0)})$.

But for low saturation the function $\mathrm{B_{eq}}(\omega)$ is convex downwards, and hence the characteristics intersect, which leads to the formation of a strong discontinuity.[17] In the neighbourhood of the discontinuity the gradients of the solution are large, and hence another expansion is necessary. To construct such a solution we will introduce the following variables

$$T = t/\varepsilon,\quad X = (x-Dt)/\varepsilon$$

which “stretches” the time and the space in the direction perpendicular to the front $x = Dt$. Equations (5.12) can then be written in the form ($\eta = 1/\tau$)

$$\frac{\partial\omega}{\partial T}+[B'(\omega)-D]\frac{\partial\omega}{\partial X}-\frac{\partial}{\partial X}\left[a(\omega)\frac{\partial\omega}{\partial X}\right] = \frac{\alpha\eta\varepsilon}{M}\Lambda(\omega,\xi),\quad \frac{\partial\xi}{\partial T}-D\frac{\partial\xi}{\partial X} = \eta\varepsilon\Lambda(\omega,\xi) \tag{5.17}$$

When $\varepsilon \to 0$ system (5.17) is defined in the half-plane $\{T \ge 0, -\infty < X < \infty\}$. Condition (5.13) serves as the initial condition. At the boundary $X \to +\infty$ the boundary condition has the form

$$\omega(T,X) = \omega_0,\quad 0 \le T \le T_0,\quad X \to +\infty \tag{5.18}$$

At the boundary $X \to -\infty$ the boundary condition is the condition for matching with the continuous solution of Eq. (5.16)

$$\omega(T,X) = \omega_D^{(0)},\quad T \ge 0,\quad X \to -\infty \tag{5.19}$$

We will now consider the steady solution of system (5.17). In this case, from system (5.17) we obtain the equation

$$\frac{dB(\omega)}{dX}-D\frac{d}{dX}\left(\omega-\frac{\alpha}{M}\xi\right) = \frac{d}{dX}\left[a(\omega)\frac{d\omega}{dX}\right]$$

Integrating it over X and taking boundary conditions (5.18) and (5.19) into account, which give $d\omega/dX \to 0$ as $X \to \pm\infty$, we obtain

$$\frac{d\omega}{dX} = \Delta(\omega, \xi), \quad \frac{d\xi}{dX} = -\frac{\eta\varepsilon}{D}\Lambda(\omega, \xi)$$

$$\Delta(\omega, \xi) = \frac{1}{a(\omega)}\left[B(\omega) - B(\omega_0) - D(\omega - \omega_0) + \frac{\alpha D}{M}(\xi - \xi_0)\right]$$

$$D = \frac{B(\omega_D^{(0)}) - B(\omega_0)}{\omega_D^{(0)} - \omega_0 - \alpha(\xi_D^{(0)} - \xi_0)/M} \tag{5.20}$$

Considering the effective saturation ω as the independent variable and taking into account the fact that system (5.20) is insensitive to a constant shift along the X axis, it can be seen that the structure of the front is given by the relation

$$X(\omega) = \int_{\omega_0}^{\omega} \frac{ds}{\Delta(s, \xi(s))} \tag{5.21}$$

The relation ξ(s) is the solution of the initial problem

$$\frac{d\xi}{d\omega} = -\frac{\eta\varepsilon}{D}\frac{\Lambda(\omega, \xi)}{\Delta(\omega, \xi)}, \quad \xi(\omega_0) = \xi_0 \tag{5.22}$$

in the interval (ω_0, ω_D).

Boundary-value problem (5.21)-(5.22), in general, requires a numerical solution.

Acknowledgements

I wish to thank I. G. Goryacheva for useful discussions.

This research was supported by the Programme of the Department of Power Engineering, Machine Construction, Mechanics and Control Processes of the Russian Academy of Sciences "Accumulation of Damage, Fracture, Wear and Structural Changes of Materials under Intensive Mechanical, Temperature and Radiation Actions" (13-OE), the Russian Foundation for Basic Research (06-05-64924) and the Programme of the Ministry of Education and Science of the Russian Federation "The Development of the Scientific Potential of Higher Schools" (111.06).

References

1. Muskat M, Meres MW. The flow of heterogeneous fluids through porous media. *Physics* 1936;**7**(9):346–63.
2. Leverett MC. Capillary behavior in porous solids Trans. *AIME* 1941;**142**:152–69.
3. Leibenzon LS. *Flow of Natural Liquids and Gases in a Porous Medium*. Moscow, Leningrad: Gostekhizdat; 1947.
4. Coussy O. *Poromechanics*. New York: Wiley; 2004. p. 315.
5. Barenblatt GI, Entov VM, Ryzhik VM. *Flow of Fluids through Natural Rocks*. Dordrecht: Kluwer; 1984.
6. Alishayev MG, Mirzadzhanzade AKh. The consideration of retardation phenomena in seepage theory. *Izv Vuzov Neft' i Gaz* 1975;**6**:71–4.
7. Khasanov MN, Bulgakova GI. Non-linear and Non-equilibrium Effects in Rheologically Complex Media. Moscow: Izhevsk: Inst Komp Issled; 2003.
8. Kondaurov VI. The thermodynamically consistent equations of a saturated thermoelastic porous medium. *Prikl Mat Mekh* 2007;**71**(4):616–35.
9. Kondaurov VI, Fortov VYe. *Fundamentals of the Thermomechanics of a Condensed Medium*. Moscow: Izd MFTI; 2002.
10. Kondaurov VI. *The Mechanics and Thermodynamics of a Saturated Porous Medium*. Moscow: MFTI; 2007.
11. Nigmatulin RI. *Principles of the Mechanics of Heterogeneous Media*. Moscow: Nauka; 1978.
12. Trusdell C. *A First Course in Rational Continuum Mechanics*. Baltimore: The Johns Hopkins Univ.; 1972.
13. Sedov LI. *Mechanics of Continuous Media*. Hackensack, NJ: World Scientific; 1997.
14. Terzaghi K. *Theoretical Soil Mechanics*. New York: Wiley; 1944.
15. Saffman PG, Taylor GI. The penetration of a fluid into a porous medoum or Hele-Shaw cell containing a more viscous fluid. *Proc Roy Soc London Ser A* 1958;**245**(1242):312–29.
16. Homsy G. Viscous fingering in porous media. *Ann Rev Fluid Mech* 1987;**19**:271–311.
17. Kulikovskii AG, Pogorelov NV, Semenov AYu. *Mathematical Aspects of Numerical Solution of Hyperbolic Systems of Equations*. Boca Raton, FL: Chapman & Hall/CRC; 2001.

Translated by R.C.G.